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Full Length Research Paper

The Heuristic Principle of Inability with an Application on the Set Theoretical Linear Continuum

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ABSTRACT

This paper proposes a mathematical generalization of certain epistemological inabilities with heuristic implications for philosophy in general and mathematical set theory specifically. With "*The world is real. But not reality.*" constituting the ontological commitment and "*everything is number*" being the enforced working hypothesis throughout the here presented, a *heuristic principle of inability* is postulated and further abstracted in set theoretical terms with a variation of the *Cantor set*. The derived properties of this variation in conjunction with the application of the heuristic principle of inability yield new aspects of *reality* which eventually motivate an *axiom of reality* based on a *single abstractum per definitionem*.

Key words: epistemology, abstract set theory, linear continuum, cantor set, heuristic, real, reality, everything is number, one

INTRODUCTION

"*The world is real. But not reality.*" shall serve as our ontological commitment for the here presented conjectures. It reflects epistemological limits with a heuristic approach on the non-increasable absolute adjective "real".

In order to orchestrate the heuristics, we associate any axiom with a limit. Whatever is not deducible out of it or whatever is in contradiction to it is deemed unreachable. Nevertheless, this principle can be inverted: If a formal system reaches limits in its specific field of application which cannot be overcome, independent of the degree of effort spent, it may pave the way for new solutions, to knowledge increase, and eventually to new axioms.

In a methodological sense, the heuristic principle of inability defines any problem by its solution, so that in many cases it may be that the original problem will be altered by the solution, i.e., that the original problem will be redefined by a new axiom.

Picturing the tremendous efforts by countless inventors, e.g., to design a mechanism, which was intended to work energetically self-sufficient, provides with a well-established illustration of the heuristic principle of inability [cf. Born 1962]:

Eventually, the failure to succeed with a *perpetuum mobile* was formalized and represents a fundamental physical law today, namely the *law of conservation of energy*. [cf. Whittaker 1949] [1]

Analogously, the *second law of thermodynamics* was derived from the inability to transform thermal energy

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[1] *Whittaker* called it "*postulates of impotence*"

without energetic effort, i.e., without work. Unlike the first law of thermodynamics, this second law of thermodynamics doesn't represent a conservation of a physical value, but rather states that a certain value called "*entropy*" is constant, i.e., *irreversible*.

With a logic criticism of the concept of *simultaneousness* as well as of the inability to observe the postulated *ether* as the carrier for electromagnetic phenomena, the well-established *Theory of relativity* provides another illustrative example of the heuristic principle of inability:

While it was state-of-the-art in the scientific community to try to materialize the *ether* through various empirical measures, all efforts to do so failed persistently [Michelson and Morley 1887]. Eventually, *Newtonian mechanics* were in course of modification by physicists like *Hendrik Antoon Lorentz* (1853-1928) and *Poincaré* (1854-1912) which proved successful to a certain extent.

In this contest it was *Einstein* (1879-1955) who declared the inability to materialize an *ether* as a principle by assuming the speed of light to be constant and independent of any state of motion [Einstein 1905]. Consequently, the physical concept of simultaneousness needed to be amended in order to avoid the *circulus vitiosus,* which originated by the fact, that an empirical establishment of the speed of light requires the value of the latter beforehand to synchronize the very clocks used for the measurement.

As a matter of fact, the original problem of the *ether* was replaced by a new view of physical dynamics.

Together with the impossibility to distinguish between *inertial* and *gravitational* mass properties of *Newtonian mechanics*, the *Equivalence principle* of *Einsteins's General relativity* and the new *relativistic dynamics* led to a deep revision and increased interdependency of the concepts of space, time, matter, energy, and electromagnetism.

In the same time, the emergence of *quantum mechanics* shed a different light on determinism than it was previously anticipated. It may serve as prime-example of our heuristic principle of inability:

Before the discovery of the *Uncertainty principle* [Heisenberg 1927], it was assumed that physical values such as *location coordinate* and *motion quantity* could at least theoretically be measured to whatever degree the spectrum of real numbers $\mathbb{R}$ would allow to, i.e., with infinite precision in terms of decimals. In contrast in turned out, that the location coordinate is always conjugated to the motion quantity in a sense that the precision of measurement remains always below an absolute limit, namely the *Planck constant* h yielding the aspect of *complementarity* which will be introduced in the final part of this paper.

But as in the previous historic examples, setting the conjugation with its absolute limit as principle not only limited the scope of the deterministic paradigm, but led physics to elaborate and control the mechanics of measurement sensitive objects as observed in atomic and subatomic scales, i.e., the field of application and the associate knowledge grew.

EPISTEMOLOGICAL PART

In order to establish a sound philosophical framework for the introduction of the heuristic principle of inability, it is indicated to sum-up the epistemological essence laid out in the historic examples:

Firstly, there is a fundamental inability for an axiom or set of axioms to cope with certain phenomena or paradigms.

Secondly, this inability is set as a principle, i.e., as a new axiom.

Thirdly, the new axiom redefines the problem where the inability first applied.

As a next step, we increase the level of abstraction for this epistemological process as a whole by only investigating the formal aspect of it, i.e., by analyzing how language is projected to the objects of our imagination and perception.
In Figure 1, *N* represents natural language and *A* axiomatized (or formalized) language (such as mathematics). The languages are symbolized in "boxes" to express their actual finiteness in terms of symbols and grammar. The axiomatized language *A* is symbolically a subset of *N* because it is thought to be less expressive than natural language, i.e., natural language generally acts as the meta-language of axiomatized languages.

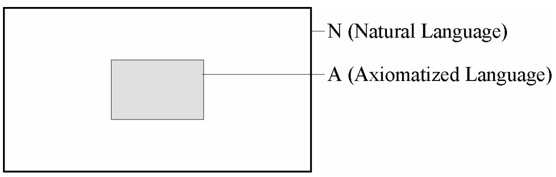

Figure 1

Being equipped with language, we project (*P*) it to objects of our imagination and perception, symbolized as reality (*R*) [Figure 2]:

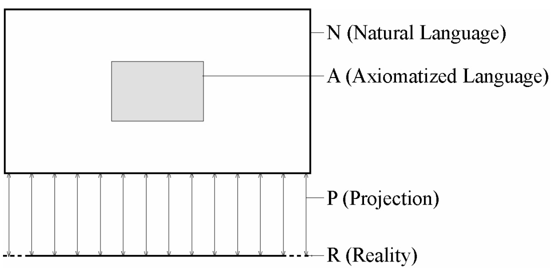

Figure 2

This projection is critical in multiple aspects with the circularity yielding an infinite regress being obvious: Any object of perception or imagination requires a corresponding term in language and *vice versa*, we only perceive or imagine reality in terms of our language capabilities (the projection (*P*) is reciprocal)[2].
According to our ontological commitment, the by far most critical aspect however seems to be represented by the fact of applying language to objects of our imagination and perception at all: *The act of projecting language to reality (and vice versa, reality to language) necessarily reduces reality to the circularity of our language and perceptive capabilities, whether axiomatized, instrument assisted, or not.*
Or, as *Azzouni* recently pointed out: "[…] what resources are available to argue for a criterion for what exists? Philosophers typically employ ontic intuitions, methodological claims, and (sometimes) descriptions of scientific practice in their philosophical arguments for one or another criterion. Establishing that argumentation for any such criterion always yields indeterminate fruit, therefore, might seem to require an analysis (or at least a survey) of more than two thousand years of metaphysical thought." [Azzouni 2010]

In order to further strengthen and explore the conjecture of the principle inability without discussing the various ontological and semantic concepts, we will investigate the most abstract and simple *objects* of our language, namely the mathematical linear continuum and

[2] An exception is constituted by *meditation* where any language and affects are kindly released. Consequently, this meditative aspect of perception cannot be communicated in any language.

its constituents (*points* or *numbers*) [Figure 3]:

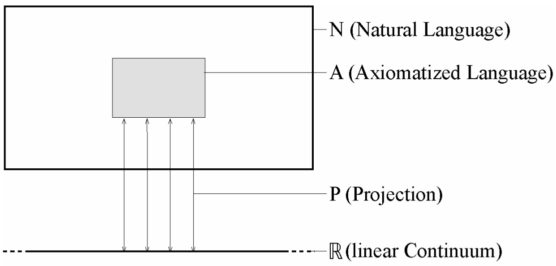

Figure 3

Applying the *Pythagorean* statement "*everything is number*", our working hypothesis essentially consists of substituting the term "reality" with the continuum while preserving our language tools (natural + axiomatized) and defining projections as *bijective* functions (one-to-one correspondences).
It shall enable us to focus on abstract processes of projections in sufficiently defined terms (if ever) as well as on a strict formal discussion where any object of our "reality" is projected to just numbers or points.
If we can provide formal evidence that the establishment of a single point and the associated linear continuum of points represents a fundamental inability, it is assumed to have a strong argument for having formalized our heuristic principle of inability in a most general sense. The heuristic aspect, i.e., a new set theoretical meta-axiom along with some empirical leads, shall constitute the concluding part of this paper.

FORMAL PART

Following the most influential analysis of the foundations of set theory to connect discrete points or numbers to a continuum, *Abraham Fraenkel* (1891-1965), *Yehoshua Bar-Hillel* (1915-1975) and *Azriel Levy* (1934-) concluded that "Bridging the gap between the domains of discreteness and continuity, or between arithmetic and geometry is a central, presumably even the central problem of the foundations of mathematics." [Fraenkel 1973].
This debate can be traced back as far as to the *Eleatic* philosophers such as *Parmenides* (515 B.C.), and *Zeno* (460 B.C.) [cf. Stokes 1971].
Putting a time-stamp on the reinitiation of the whole discussion, *Karl Weierstrass* (1815-1897) could be regarded as the father of modern analysis being the first to come up with a complete arithmetization of mathematical analysis. To do so, *Weierstrass* defined a positive real number to be a countable set of positive rational numbers for which the sum of any finite subset always remains below a pre-assigned bound. Eventually, he broke down conditions which would enable a comparison of two such “real numbers” in terms of equality or magnitude (strictly smaller than one another) [cf. Bell 2010].
But the most revolutionary thinker in contemporary history was *Georg Cantor* (1845-1918). His view of the continuum as infinite point sets laid the foundations of his theory of transfinite numbers. From there on, the geometric origin of the continuum as a collection of points was transferred to the current concept of general abstract set theory.
Just like *Weierstrass* and *Richard Dedekind* (1831-1916), *Cantor* intended to provide a definition of real numbers which avoids their a priori existence. To do so, *Cantor* looked at rational numbers and following *Cauchy* (1789-1857), he called a sequence of rational numbers a_1, a_2,…, a_n,… a *fundamental sequence* if there exists an integer N such that, for any positive rational ε, there exists an integer N, such that $|a_{n+m} - a_n| < \varepsilon$ for all m and all $n > N$. Any sequence $< a_n >$ which satisfies this condition is then said to have a *definite limit b*. *Dedekind* interpreted irrational numbers as "mental

objects" associated with cuts (*Dedekind cuts*). In analogy, *Cantor* regarded these well-defined limits as *symbols* which represent *fundamental sequences* (hereafter *Weiserstrass-Dedekind-Cantor program*). Accordingly, the domain *B* of limit points are considered an enlargement of the domain *A* of rational numbers, i.e., representing real numbers. *Cantor* showed that every single point of the line corresponds to a definite element of the domain *B* while each element of *B* determines a definite point on the line. Without providing proof of this intuitive property of the continuum, *Cantor* introduced it as an axiom, just as *Dedekind* had done with his own *principle of continuity*. We will get back to this constitutive axiom further down.

While prior *Cantor's* work the continuum has essentially been regarded as an unanalyzable concept, *Cantor* gave it an arithmetic framework. With this at hand, he identified the set of points thought as the linear continuum with numbers, enabling the comparison of "sizes" of point sets with the well established concept of *unambiguous bijections* [Figure 4 and 5]:

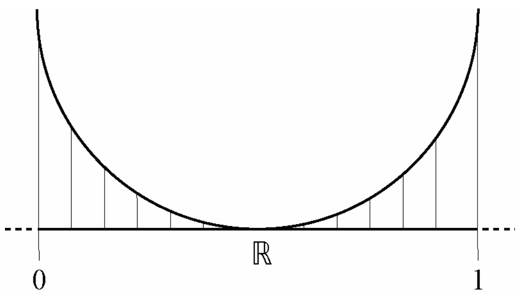

Figure 4 Every point of the finite unit interval [0;1] which is part of the infinite linear continuum ℝ, has a *bijective* projection to the points of the semi-circle (1:1 correspondence).

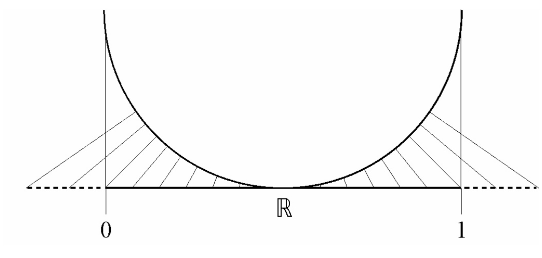

Figure 5 Analogously, every point of the infinite linear continuum ℝ has a *bijective* projection to the points of the semi-circle (1:1 correspondence), hence, the number of points of the finite unit interval [0;1] is the same as of the total linear continuum ℝ (again a 1:1 correspondence).

Cantor not only showed that a finite unit interval [0;1] of an infinite line has the same *cardinality* (*magnitude* or *power* in terms of *number of elements*) as the infinite line itself, but he also generalized this finding and showed, that all spaces E^n have the same cardinality as the set of real numbers in the one dimensional, finite unit interval [0;1]. Eventually, *Cantor* stated the hypothesis (*Continuum Hypotheses* or CH), that any infinite point set has either the cardinality of the set of natural numbers ℕ which are denumerable, i.e., $\aleph_0$ (aleph zero), or that of the non-denumerable unit interval [0;1] of the real line, i.e., $2^{\aleph_0} = C$ which has the next highest cardinality $\aleph_1 = C$.

Referring to the definition of real numbers in terms of fundamental sequences, *Cantor* introduced the *Euclidean n-space* E^n as the set of all *n*-tuples of real numbers $<x_1,x_2,\dots,x_n>$, calling each such an *arithmetical point* of E^n. The distance between two such points is represented by

$$\sqrt{\left(x_1' - x_1\right)^2 + \left(x_2' - x_2\right)^2 + \dots + \left(x_n' - x_n\right)^2}$$

Eventually, an arithmetical point-set in E^n is any point-aggregate of the points of the *Euclidean n-space* E^n.

Having singled out E^n as the analytical framework of the linear continuum, *Cantor* defined the derived set of a point set *P* to be the set of *limit points* of *P*, where a limit point of *P* is a point of *P* with infinitely many points of *P* arbitrarily close to it [Cantor 1932: 140].

He called a point set *perfect* if it coincides with its derived set. *Cantor* himself realized that the condition of "perfectness" is insufficient to characterize the intended, intuitive continuum. He noticed in a footnote [cf. Cantor 1932: 207] that one could construct perfect sets which are just nowhere dense in any interval of the linear continuum. Today it is coined "*Cantor set*" or "*Cantor ternary set*" and constitutes an illustrative argument in furtherance of our heuristic principle of inability:
Following *Cantor*, a perfect, nowhere dense set in any closed interval of the linear continuum ℝ is defined as real numbers such as:

Def. 1 $$x = \frac{c_1}{3} + \ldots + \frac{c_v}{3^v} + \ldots$$

where c_v can be regarded as having the values 0 or 2 for each integer *v*. [cf. Fleron 1994]

Def. 2 A set *S* is perfect if $S = S'$, where S' is the set of all the limit points of *S*.

Def. 3 A set *S* is nowhere dense if the interior of the closure of *S* is empty.

Cantor introduced this set as simply as:

Let *I* be an interval [0;1]. Split *I* into thirds. Remove the open set that represents the middle third and let A_1 be the remaining set:

Def. 4 $$A_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Removing the open middle third interval from each of the two closed sets in A_1 continuously yields the remainder A_2:

Def. 5
$$A_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$
Each consecutive step *k* for $k \in \mathbb{R}$ consists of removing the open middle third interval from each of the closed sets in A_k. We call the remaining set A_{k+1}. For each $k \in \mathbb{R}$, A_k is the union of 2^k closed intervals each of *length* 3^{-k}.

Def. 6 $$C_3 = \bigcap_{k=1}^{\infty} A_k$$

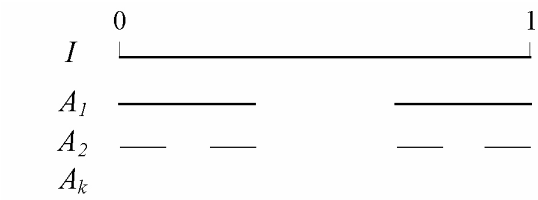

Figure 6 Consecutive removal process of the middle third interval of every closed interval leaving a single point *"extended" with maximum "disconnection" over the whole unit interval* [0;1] (cf. fn 3: 13).

The ternary (base 3) expansion of the *Cantor set C* only consists of 0s and 2s because at any step of removal, every number with a ternary expansion containing a 1 is removed. In the first step of the removal any number remaining can be viewed of having the digit c_1 = 0 or 2 where $x = 0.c_1c_2c_{3\ldots}$, because if $x \in \left[0, \frac{1}{3}\right]$, $c_1 = 0$ and if $x \in \left[\frac{2}{3}, 1\right]$, $c_1 = 2$.

For *x* remaining after *n* removals, the repetition for each step of removal yields c_n being either 0 or 2.

The constitutive properties of the *Cantor set* C_3 are severally proven [cf. Wikipedia 2012] so that we will just list them as follows and eventually get into one or the other proof, as required:

- non-denumerable (cardinality of the continuum)
- contains no intervals (all intervals were removed)
- zero lenght (*Euclidian point topology* in E^n)
- compact (countable many endpoints)
- nowhere dense (no connected points at all)
- *Hausdorff dimension* log2/log3

In order to hold on to his "intuitive" definition and not being obliged to consider constructions of perfect sets in the manner of C_3 as a continuum, *Cantor* introduced an additional condition which he called a "*connected set*" representing a metric definition: A point set T is connected in *Cantor's* sense if for any pair of its points t, t' and any arbitrarily small number ε there is a finite sequence of points $t_1, t_2, \ldots, t_n$ of T for which the distances $[tt_1]$, $[t_1t_2]$, $[t_2t_3]$, …, $[t_nt']$ are all less than ε. *Cantor* was now able to define a continuum to be a *perfect connected point set*: "Die *perfekten* Punktmengen S sind keineswegs immer in ihrem Innern das, was ich in meinen vorhin genannten Arbeiten "überalldicht" genannt habe[11)]; deshalb eignen sie sich auch noch nicht allein zur vollständigen Definition eines Punktkontinuums, wenn man auch sofort zugeben muß, dass letzteres stets eine *perfekte* Menge sein muß.
Es ist vielmehr noch ein Begriff erforderlich, um im Verein mit dem vorhergehenden das Kontinuum zu definieren, nämlich der Begriff einer *zusammenhängenden* Punktmenge T.
Wir nennen T eine *zusammenhängende* Punktmenge, wenn für je zwei Punkte t und t' derselben bei vorgegebener beliebig kleiner Zahl ε immer eine endliche Anzahl Punkte $t_1, t_2, \ldots, t_\nu$ von T auf mehrfache Art vorhanden sind, sodaß die Entfernungen tt_1, t_1t_2, t_2t_3, …, $t_\nu t'$ sämtlich kleiner sind als ε. [Es handelt sich also um eine "metrische" Eigenschaft des Kontinuums.]
Alle uns bekannten geometrischen Punktkontinua fallen nun auch, wie leicht zu sehen, unter diesen Begriff der *zusammenhängenden* Punktmenge; ich glaube aber nun auch in diesen *beiden* Prädikaten "perfekt" und "zusammenhängend" die notwendigen und *hinreichenden* Merkmale eines Punktkontinuums zu erkennen und definiere daher ein Punktkontinuum innerhalb G_n als *perfekt-zusammenhängende Menge*[12)]. Hier sind "perfekt" und "zusammenhängend" nicht bloße Worte, sondern durch die vorangegangenen Definitionen aufs schärfste begrifflich charakterisiert, ganz allgemeine Prädikate des *Kontinuums*." [Cantor 1932: 194]

As introduced earlier, both, *Dedekind* and *Cantor*, were fully aware of the axiomatic, yet not necessary assumption of any continuity within the apparently perceived, three dimensional physical space, which both however considered as *reality*: "An diese Sätze knüpfen sich die Erwägungen über die Beschaffenheit des der realen Welt, zum Zwecke begrifflicher Beschreibung und Erklärung der in ihr vorkommenden Erscheinungen, zugrunde zu legenden dreidimensionalen Raumes. Bekanntlich wird derselbe sowohl wegen der in ihm auftretenden Formen, wie auch namentlich mit Rücksicht auf die darin vor sich gehenden Bewegungen als *durchgängig stetig* angenommen. Diese letztere Annahme besteht nach den gleichzeitigen, voneinander unabhängigen Untersuchungen Dedekinds (M.s. das Schriftchen: Stetigkeit und irrationale Zahlen von R. Dedekind, Braunschweig 1872) und des Verfassers (Mathem. Annalen Bd. V, S.127 und 128) [II5, S.96] in nichts

anderem, als daß jeder Punkt, dessen Koordinaten *x, y, z* in bezug auf ein rechtwinkliges Koordinatensystem durch *irgendwelche* bestimmte reelle, rationale oder irrationale Zahlen vorgegeben sind, *als wirklich zum Raume gehörig* betrachten wird, wozu im allgemeinen kein innerer Zwang vorliegt und worin daher ein freier Akt unserer gedanklichen Konstruktionstätigkeit gesehen werden muß. Die *Hypothese der Stetigkeit des Raumes* ist also nichts anderes, als die an sich willkürliche Vorraussetzung der vollständigen, gegenseitig-eindeutigen Korrespondenz zwischen dem dreidimensionalen *rein arithmetischen Kontinuum* (*x, y, z*) und dem der Erscheinungswelt zugrunde gelegten Raume.[1]
Unser Denken kann aber mit gleicher Leichtigkeit von einzelnen Raumpunkten, sogar wenn sie überalldicht vorkommen, sehr wohl abstrahieren und sich den Begriff eines *unstetigen* dreidimensionalen Raumes von der im vorhergehenden charakterisierten Beschaffenheit bilden. Die sich alsdann ergebene Frage, ob auch in so *unstetigen* Räumen *stetige Bewegung* gedacht werden könne, muß nach dem Vorangehenden unbedingt *bejaht* werden, weil wir gezeigt haben, daß je zwei Punkte eines Gebildes durch unzählig viele stetige, vollkommen reguläre Linien verbunden werden können. Es stellt sich also merkwürdigerweise heraus, daß aus der bloßen Tatsache der stetigen Bewegung auf die durchgängige Stetigkeit des zur Erklärung der Bewegungserscheinungen gebrauchten dreidimensionalen Raumbegriffs zunächst kein Schluß gemacht werden kann. Daher liegt es nahe, den Versuch einer modifizierten, für Räume von der Beschaffenheit gültigen Mechanik zu unternehmen, um aus den Konsequenzen einer derartigen Untersuchung und aus ihrem Vergleich mit Tatsachen möglicherweise wirkliche Stützpunkte für die Hypothese der durchgängigen Stetigkeit des der Erfahrung unterzulegenden Raumbegriffs zu gewinnen." [Cantor 1932: 156-157]

To bridge the different concepts of arithmetic and geometry, *Cantor* finally needed to introduce the axiom of connecting any arithmetic value to a specific point of a line: "Daß nun ebenso auch die Zahlengrößen der Gebiete *C, D,* … befähigt sind, bekannte Entfernungen zu bestimmen, ergibt sich ohne Schwierigkeit. Um aber den in diesem § dargelegten Zusammenhang der Gebiete der in §1 definierten Zahlengröße mit der Geometrie der geraden Linie vollständig zu machen, ist nur noch ein *Axiom* hinzuzufügen, welches einfach darin besteht, daß auch umgekehrt zu jeder Zahlengröße ein bestimmter Punkt der Geraden gehört, dessen Koordinate gleich jener Zahlengröße, und zwar im dem Sinne gleich ist, wie solches in diesem § erklärt wird[1].
Ich nenne diesen Satz ein *Axiom*, weil es in seiner Natur liegt, nicht allgemein beweisbar zu sein.
Durch ihn wird denn auch nachträglich für die Zahlengrößen eine gewisse Gegenständlichkeit gewonnen, von welcher sie jedoch ganz unabhängig ist." [Cantor 1932: 97]

Having singled out the intuitive path of the *Weiserstrass-Dedekind-Cantor program* explicitly in terms of *connectivity*, *continuity*, and *arithmetization*, it is about time to account for the implicit consequences. As multiply shown, their axiomatic program intended a *complete arithmetization* along with a *geometric materialization* of the continuum.
The great idea was to combine arithmetic notions and associated values as *the*

discrete per definitionem with metric continuity.
In arithmetic, we consider those values as different which differ in any term *a*, no matter in which position $(a_1a_2a_3...a_n)$. Or else, without any exception, all arithmetic values have to be considered as equivalent with the concept of distinct numbers being obsolete. Accordingly, the distinct character of arithmetic values represents *Cantor's* key argument for the non-demunerability of the set of real numbers $\mathbb{R}$ [cf. Cantor 1991: 35; letter to *Dedekind* 7.12.1873].

In geometry, a point shares the same distinct property as long as observed isolated. Therefore, the *bijective* connection between arithmetic values and geometric points seems evident.
However, applying the *axiom of continuity* may yield to the asymmetry of a discrete but connected continuum which may neither be a necessary nor a commonly "desired" property of the latter.
Therefore, thinkers such as *Brentano (*1838-1917), *C. S. Peirce* (1839-1914)*, Poincaré*, *L. E. J. Brouwer* (1881-1966) and *Weyl* (1885-1955) to name a few, were opposed to the concept of a discrete but connected continuum. For any aggregate number of geometric points maintains zero topological dimension just as any aggregate of numbers will remain different if they differ in any decimal. If points however would constitute a dimensional object such as a continuous line, it would imply that the continuity of every point would be constituted by smaller points, and that: *ad infinitum*. But as soon as we try to constitute a geometric line with formerly isolated points connected to arithmetic values, the chasm between the discrete and continuity unfolds as deep as this ancient debate already lasts.

The crucial point to define the continuum throughout the whole debate, from the ancients to the here presented, consists of having *recognized* the constraints imposed by holding to any idea of space metric and associated topology, but having *failed to interpret* this constraint *formally*.
In an effort to deliver an appropriate formalization, we will now enforce the idea of a total arithmetization of the linear continuum. Therefore, we generalize C_3 in a way, that any topological dimension whatsoever is eliminated:
While it can easily been shown that although C_3 is has the *Lebesgue measure* of a single point of *Euclidian* dimension zero due to the removal of all one-dimensional line intervals with total length 1 [cf. Wikipedia 2012]*,* C_3 can still be associated with a so called *Hausdorff dimension* [cf. Hausdorff 1919] in a *non-Euclidean Hausdorff topology* with

$$\dim(C_3) = \frac{\log 2}{\log 3} \approx 0.63$$

A variation of deriving the *Cantor set* will provide us with *Hausdorff* $\dim(C_k) = 0$ so that we look at a non-dimensional *point* without any topologic association whatsoever as follows [cf. Falconer 1985]:

Def. 7 The dimension of the *Cantor ternary set* (C_k) is: $d = \dfrac{\log \frac{1}{2}}{\log\left(\frac{1}{2} - \frac{1}{2k}\right)}$

Def. 8 Let $\{\Phi_i\}_{i=1}^{k}$ be a collection of similarities such that $E \subseteq \mathbb{R}^n$ is invariant with respect to $\{\Phi_i\}_{i=1}^{k}$. If $\{\Phi_i\}_{i=1}^{k}$ satisfies the open set condition and r_i will be the ratio of the *i*-th similarity Φ_i, then the

Hausdorff dimension of E is equal to the unique positive numbers for which $\sum_{i=1}^{k}(r_i)^s = 1$.

Let $\{D_k\}$ now be a collection of sets defined by k for $k \geq 2$ (for k=0 or k=1 either no interval at all or the whole interval [0;1] will be removed not yielding the properties of a *Cantor set*) in which each set is build by a repetitive removal of an open interval of length $\left(1-\frac{2}{k}\right)$ from the center of each closed interval [0;1]. Only intervals of length $\frac{1}{k}$ will remain on each side. We now only vary the length of the *side intervals* in terms of k but keep the ternary property. Then we remove the interval in-between where the original method thought by *Cantor*, as already discussed, only removed the open middle third interval with no variation in terms of k, i.e., k was a constant with k=3.

Applying *Def. 7* we calculate the general *Hausdorff dimension* for any $k \geq 2$:

Let $\Phi_1(x)$ and $\Phi_2(x)$ be:

$$\Phi_1(x) = \left(\frac{1}{k}\right)x \text{ and } \Phi_2(x) = \left(\frac{1}{k}\right)x + 1 - \frac{1}{k}. \quad (1.0)$$

With $C_k = \bigcup_{i=1}^{2}\Phi_i(C_k)$ and $r_1 = \frac{1}{k}$ as well as similarly $r_2 = \frac{1}{k}$ we resolve s such that

$$\sum_{i=1}^{2}(r_i)^s = 1 \quad (1.1)$$

and

$$2\left(\frac{1}{k}\right)^s = 1 \text{ iff } s = \frac{\log 2}{\log k} \quad (1.2)$$

and finally:

$$\dim(C_k) = \frac{\log 2}{\log k} \quad (1.3)$$

where

$$\lim_{k\to\infty}\dim(C_k) = \lim_{k\to\infty}\left(\frac{\log 2}{\log k}\right) = 0 \quad (1.4)$$

As intended, we have a set C_k with *Cantor set* properties with

$$\dim(C_k) = 0, \quad (1.5)$$

i.e., that of a point of a *n-space* E^n without any topology just by dividing the interval [0;1] into three subintervals where the smaller the length of the side intervals $\frac{1}{k}$, *viz.*, the larger the removed length of the interval $\left(1-\frac{2}{k}\right)$, the more the *Hausdorff dimension* tends to 0.

Proposition 1 Any point of an arbitrary *n-space* E^n has the cardinality of the linear continuum $\mathbb{R}$.
Having provided C_k as a *point* of the *n-space* E^n without any topology but with *Cantor set* properties, it can been shown that the cardinality of a *single point* is equal to that of the linear continuum $\mathbb{R}$:

Proof For each step of the repetitive removal process of an open interval of length $\left(1-\frac{2}{k}\right)$ from the center of each closed interval [0;1] we can again combine a binary with a ternary notation as follows:
Since every closed interval only has one open one removed, we can look at C_k of having a ternary expansion. We define a

function $f(x)$ onto the unit interval [0;1] and write in base 3 for every $x \in C_k$ either "0" or "2" without using the digit "1". Consequently, $f(x)$ is the point in the closed interval [0;1] whose binary expansion is obtained by substituting each digit "2" in the ternary expansion of x by the digit "1". Eventually, all points of the unit interval [0;1] can be derived by this process while we already know about the unit interval [0;1] having the cardinality of the linear continuum ℝ. We know further, that any arbitrary *n-space* E^n has the cardinality of ℝ. Hence, the cardinality of a *point* is equal to that of the linear continuum ℝ.

Apparently, our concept of a single point requires some further analysis. If we hold to the axiom of connecting any arithmetic value to a specific point of a line, we would have to assign the value 1 to our *single point* C_k.[3]

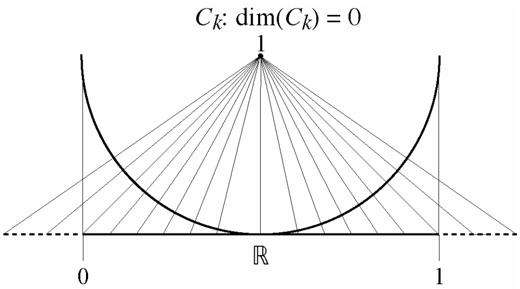

Figure 7 Every point of the infinite linear continuum ℝ has a *bijective* projection to the points of C_k. While C_k is represented by a *single point* of the arbitrary *n-space* E^n without any topology defined thereon denoted with "1", it has the same cardinality as the whole linear continuum ℝ.

If this conjecture was not obvious with C_3 which still implied a topological dimension $\dim(C_3) = \frac{\log 2}{\log 3} \approx 0.63$, it became inevitably with C_k.

Stepping back to the very conception of the *Cantor set*, the supposed *paradox* of finding an equivalence between the cardinality of the linear continuum and a single point may result from missing clarity in the underlying assumptions of the very construction of the *Cantor set* with regard to *which* axioms are applied:

From a mere arithmetic perspective the apparent paradox is less obvious since we know that arithmetic values *per se* do not "occupy" any space which we consider as *physically* real. It is only the axiom of connecting arithmetic values to geometric points which may give rise to an apparent paradox, for a geometric line induces much more association to any space which we consider as physically real than just arithmetic values which may only occupy a designated, *imaginary arithmetic space* at the most.

But even this axiom is not yielding any paradox because as long as we do not associate a convention about a *physical distance* between any two distinct points (*A, B*) or arithmetic values (*x, y*), i.e., a *length* to any aggregate of points of a geometrical line, we can indeed "occupy" *any imaginable* number of points or arithmetic values in a non-spatial entity such as in a single point.

Cantor's concern was about defining a continuum to be continuous. He *only* defined an *Euclidean* standard metric in *arithmetic terms* onto the closed unit interval [0;1] where the "interval removal process" *only* demonstrates the whole interval being left as a point but with the same number of points as the initial unit interval [0;1] with the induced topology of the *Cantor set* being maximally disconnected, i.e., discrete vs. continuous.

However, this demonstration neither serves the purpose of holding to an idea

[3] Note in course of correction: further detailed consideration suggests E^n and the single integer point to be omni-equivalent.

of an *extended* nor of a *continuous* reality because with any appropriate number of points or numbers (hereafter referred to as *elements*), we can just define any metric we want onto the elements, discrete or connected, and that is *independent of any physical reality*, extended or not.

Therefore, the next important axiom enabling to hold to such an *extended*, *continuous* aspect of reality is the correspondence between a *physical entity* which is a *convention* about a *physical length* and between a *metric distance* of two elements.

Unless we admit physical space with zero spatial extension to our imagination which we will enforce in furtherance of this paper, a *Cantor set cannot be constructed physically* where the smallest theoretical length with physical yet not directly observable property would be the *Planck length* ℓ_p [cf. Wikipedia 2012; 2]. It is defined by three fundamental physical constants, namely by the speed of light in vacuum *C*, the reduced *Planck constant* $\hbar$, and the gravitational constant *G* as:

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616199 \times 10^{-35}\mathrm{m}$$

where ℓ_p is a direct consequence of quantum mechanical measurement process which is restricted by *Heisenberg's uncertainty principle* as introduced earlier. Although the *Planck length* is a physical unit defining a discrete space metric, it is only about 10^{-20} of the diameter of a "proton" and thereby orders of magnitudes smaller than today's precision of measurement.

It is in this sense, that C_3 and C_k so far only prove that the space which we consider as physically real requires a convention about a metric system such as the *International System of Units* where among others, a length is a reference to e.g., light. Eventually, a *length unit* can be arbitrarily defined as the length of the path travelled by light in vacuum in a fixed, finite time interval [cf. Wikipedia 2012; 3].

Nevertheless, the most important axiom to explicitly observe when considering elements of our imagination or conventions about what we consider as physically real, is to connect the latter to *reality*.

Hence, equivalent to *proposition 1* would be the statement of a constant function *f* defined over all real numbers $\mathbb{R}$ having the value 1, i.e., $f(x) = 1 \quad \forall x \in \mathbb{R}$, where the projection of $\mathbb{R}$ is *surjective* onto 1 and with a single point being equivalent to C_k in terms of number of elements being *surjective* onto $\mathbb{R}$, i.e., having a *bijective* identity function $f(x) = x \quad \forall x \in \mathbb{R}$. But the identity can only hold if 1 is redefined in set theoretical terms being as well as both, a single unit *and* a multitude. It is therefore suggested, to shift the debate from the characteristics axiomatically and "intuitively" defined into the idea of an extended linear continuum to a closer look on the properties of 1.

To do so, we reconsider the *compact, perfect metric* of C_k and its *totally disconnected, discrete topology* which was inherited from the *Euclidean* standard metric *arbitrarily defined onto* the closed unit interval [0;1]. And just as having arbitrarily defined the *Euclidean* standard metric onto the closed unit interval [0;1], we arbitrarily *abstract* any metric property away from C_k as we previously did with its *Hausdorff topology* with just non-denumerable elements remaining as a *single point*.

Now we can consider the following proposition:

Lemma 1 1 has at least the cardinality κ of C_k with $|C_k| \leq |1|$.

Proof Since we know that 1 has at least the cardinality κ of C_k, we only need to show that the power set of C_k is at least equal to 1, i.e., $\mathcal{P}(C_k) \leq 1$.

To do so, we apply *Cantor's theorem*:
For any set A, the set of all subsets of $\mathcal{P}(A)$ has a strictly greater cardinality than A itself with $|\mathcal{P}(A)| > |A|$.

While the number of points of $\mathcal{P}(C_k)$ are strictly greater than C_k, $\mathcal{P}(C_k)$ has any metric and topology abstracted away, hence still remaining the single point 1.

Proposition 2 1 is at least equal to any cardinality κ_x with $\boldsymbol{\kappa}_x \leq |1|$.

Proof Since we know that 1 has at least the cardinality κ of C_k, we only need to show that there is always a subset A of C_k that is smaller than C_k, i.e., $A \subseteq P(A)$.

To do so, we apply the inversion of *Cantor's theorem*:
For any power set $\mathcal{P}(A)$, the set A is strictly smaller than $\mathcal{P}(A)$ with $|A| < |\mathcal{P}(A)|$.
Obviously, with a recursion on $|\mathcal{P}(A)| = |2^A|$ we can generate greater and greater power sets of any given set while the number of points of the ever-increasing power sets are always less or equal to the single point.

We reconnect this finding to *Cantor's* original view of magnitudes when he defined a cardinal number being the result of a *double abstraction* in the following sense:

Right after defining a set as *a gathering into a whole of definite, distinct objects of our perception and of our thought*, which *Cantor* called *elements of a set*, he describes cardinality as an abstraction from the nature of the elements *m* of a set *M*. With some order may being left within and among these elements *m*, one abstracts even from that order to compare the magnitude of any two sets *M* and *N* where the elements itselves cannot be distinguished any further all becoming "one" as an intellectual image or projection of any given set *M* existing in our spirit. For any set *N* that is *bijective* with *M* the cardinal number would eventually be the same, i.e., 1:
"*‚Mächtigkeit' oder ‚Kardinalzahl' von M nennen wir den Allgemeinbegriff, welcher mit Hilfe unseres aktiven Denkvermögens dadurch aus der Menge M hervorgeht, daß von der Beschaffenheit ihrer verschiedenen Elemente m und von der Ordnung ihres Gegebenseins abstrahiert wird.*
Das Resultat dieses zweifachen Abstraktionsakts, die Kardinalzahl oder Mächtigkeit von *M*, bezeichnen wir mit

$$\overline{\overline{M}}\,. \qquad (3)$$

Da aus jedem einzelnen Elemente *m*, wenn man von seiner Beschaffenheit absieht, eine ‚Eins' wird, so ist die Kardinalzahl *M* selbst eine aus lauter Einsen zusammengesetzte Menge, die als intellektuelles Abbild oder Projektion der gegebenen Menge *M* in unserem Geiste Existenz hat." [Cantor 1932: 282-283]

With that, the collection of all sets *N* with $|N| = |M|$ exists *ad infinitum* where for every *x,* the set $\{x\}$ has exactly one element $\{1\}$ with cardinality 1 so that the double abstraction leaves an object without any specific properties other than *existing*.
If now, according to *Cantor*, a well determined, *finished* set would have a

cardinality which would not correspond to any aleph, it would need to include subsets whose cardinality is *any* of the alephs, i.e., this set would need to carry the totality of alephs within itself: ",Wenn eine bestimmte wohldefinierte *fertige* Menge eine Cardinalzahl haben würde, die mit keinem der Alefs zusammenfiele, so müßte sie Theilmengen enthalten, deren Cardinalzahl *irgend* ein Alef ist, oder mit anderen Worten, die Menge müßte die Totalität aller Alefs in sich tragen.'" [Cantor 1991: 388; letter to *Hilbert* 26.9.1897]

In furtherance of this idea, we find 1 being *finite* with its ordinal number α and cardinal number κ being identical, i.e., 1. Hence, 1 is not any transfinite cardinal $\aleph$. However, *proposition 2* showed that $|1| \geq$ any $\boldsymbol{\kappa}$ which implies that 1 is *constant* and *at least equal* to the cardinal number $\aleph_0$ of $\mathbb{N}$, while being *always greater* than any $\aleph$:

Def. 9

$$\Omega = \{\alpha | \alpha \text{ is an ordinal number}\}$$

where for all ordinal numbers α there is an ordinal number β such that

$$|W(\alpha)| < |W(\beta)|.$$

Reinstating *Cantor's* question if there is a system ת (taw)
$[\text{of all alephs } \aleph_0, \aleph_1, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots]$
of *all transfinite cardinal numbers* that is not an aleph [cf. Cantor 1991: 410; letter to *Dedekind* 3.8.1899] yields:

Proposition 3 $\{\aleph_\alpha | \alpha \in \Omega\} < 1 = ת$

Proof ת shall be a set with $\aleph^*$ being the *supremum* of a set of cardinal numbers $\aleph$ with $\aleph^* = \sup$ (ת).
Since the *supremum* of a set of cardinal numbers is itself a cardinal number, we have $\aleph^* \in$ ת.
While ת contains all $\aleph^*$ with $\kappa \in$ ת, hence ת not containing a greatest element and $\kappa = \aleph_\alpha$ so that $\kappa < \aleph_{\alpha+1}$, ת remains constantly 1 with $\kappa < \aleph_{\alpha+1} < |1|$.

With ת = 1 we apparently have a finished set (just as any aleph is considered to be a finished set) that has a cardinality which does not correspond to any aleph and which includes subsets whose cardinality is indeed *any* of the alephs, i.e., ת carries the totality of alephs within itself. However, the all-imposing question to be answered is whether ת is a *consistent set*.
Again with *Cantor* we find that if we consider any *finite* multitude to be consistent, we can extend this attribute to any *transfinite* multitude as represented by the alephs. And just as the consistency of any finite multitude solely depends on the unprovable axiom of arithmetic with 1+1 being 2, so is the extension to the transfinite arithmetic with its cardinal numbers represented by alephs:
"Man muß die Frage aufwerfen, woher ich denn wisse, daß die wohlgeordneten Vielheiten oder Folgen, denen ich die Cardinalzahlen $\aleph_0, \aleph_1, \ldots, \aleph_{\omega_0}, \ldots, \aleph_{\omega_1}, \ldots$ zuschreibe, auch wirklich ‚Mengen' in dem erklärten Sinne des Wortes, d.h. ‚konsistente Vielheiten' seien. Wäre es nicht denkbar, daß schon *diese* Vielheiten ‚inkonsistent' seien, und daß der Widerspruch der Annahme eines ‚Zusammenseins aller ihrer Elemente' sich *nur noch nicht bemerkbar gemacht hätte*? Meine Antwort hierauf ist, daß diese Frage auf *endliche Vielheiten ebenfalls auszudehnen* ist und daß eine genaue Erwägung zu dem Resultate

führt: sogar für endliche Vielheiten ist ein ‚Beweis' für ihre ‚Consistenz' nicht zu führen. Mit anderen Worten: Die Thatsache der ‚Consistenz' endlicher Vielheiten ist eine einfache unbeweisbare Wahrheit, es ist ‚*Das Axiom* der Arithmetik (im alten Sinne des Wortes)'. Und ebenso ist die ‚Consistenz' der Vielheiten, denen ich die Alephs als Cardinalzahlen zuspreche, ‚das Axiom der erweiterten, der transfiniten Arithmetik.' ..." [Cantor 1991: 412; letter to *Dedekind* 28.8.1899].

Accordingly, we have to consider ת = 1 to be a consistent multitude while at the same time, 1 is evidently finite and represents *absolute unity* being *the most common denominator*, literally in terms of arithmetic and figuratively in terms of a geometric point. There seems to be no other conclusion than considering 1 itself representing an *inconsistency at the very foundation of mathematics*.

CONCLUSION

It is evident, that continued abstraction eventually yields to some concept of unity which is necessarily *complementary*, i.e., incorporates properties that may appear to be contradictory: While *Cantor* shed light on entities thought to be transfinite, unifying the concepts of finiteness with the infinite, e.g., with the finite unit interval [0;1] on the linear continuum having the same number of elements as the whole infinite linear continuum itself, the next level of abstraction merges both concepts to the most abstract notion, i.e., to the unit of a point and 1:

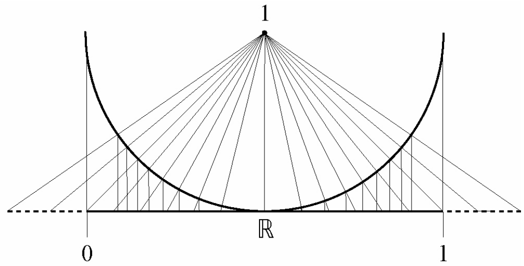

Figure 8 Continued *abstraction* unified the concepts of finiteness with transfiniteness and eventually with unity where the non-denumerable number of elements of the closed unit interval [0;1] of the non-denumerable number of elements of the real line ℝ are identical whereas 1 as single unit incorporates all of these non-denumerable elements and all possible power sets thereof including all subset of ℝ.

And no matter if we opt for the arithmetical language saying that "*everything is number*" or if we follow the geometrical track with *points* to express the projections of our imagination, at a certain stage of abstraction we have to account for the undefined or indefinable constituents of our *reality*. Otherwise our intuition about reality may be misled by the circularity imposed by unreflected language as introduced earlier in this paper.

To finally formalize our heuristic principle of inability, we implement our epistemological scheme and consider 1, a point, or the very notion "*something exists*" as the *most fundamental inability* of our language in terms of an inability of determination.

It is not surprising, that *Cantor* himself was very well versed in this essential constraint of language when he observed as well as agreed with *Spinoza* (1632-1677), that "*omnis determinatio est negatio"* [Cantor 1932: 175].

Next, we set the inability of determination as a principle where we define the *abstractum per definitionem* by inversion: "*omnis negatio est determinatio*", which is equivalent to the complementary expression that *total negation yields something* or *negation of* *totality yields something* where "something" can still be "everything" *and* "a single thing".

The complementary nature of this "something" is different from what our common intuition and formal language previously suggested. We therefore substitute the *Latin* type-font for 1 with the corresponding *Arabic* font ١ leaving semantic freedom for both, *multitudes* and *diversity* as well as for *unity*, i.e., having distinguished it from a supposedly non-complementary, naïve arithmetic meaning.

A streamlined formal expression for a total negation suggests to look for an already commonly known yet *undefined expression* such as $0 \cdot \infty$ which we define as follows:

Def. 10 0 = negator
∞ = totality
١ = something

Axiom of reality $0 \cdot \infty =$ ١

Finally, we apply the new axiom and redefine the debate about the linear continuum with the world of phenomena underlying reality where the inability of defining *reality* first appeared:

With ١ not only being the most inclusive and powerful notion containing all and any elements of our thought, imagination and perception, but at the same time also being the most fundamental entity, the question of an extended, connected, divisible or non-divisible multitude or just an indivisible single entity is redefined by the question about how any kind of reality could consistently be expressed.

In line with *Cantor*, our conclusion is that it cannot. The absolute, i.e., the absolute greatest (*sive Deus*) cannot be determined by us but only by itself: "Es versteht sich von selbst, dass hierunter [*transfinite numbers*] das Absolute d.h. das absolut Größte (sive Deus) nicht zu verstehen ist, welches nur durch sich selbst, nicht aber von uns determiniert werden kann; ..." [Cantor 1991: 174; letter to *Laßwitz* 3.2.1884]

The here presented *axiom of reality* is a complementary notion where multiple, apparently contradictory properties on highest level of abstraction, i.e., the most undefined expression, yields something (١).

Figure 9 tries to illustrate this process: leaving any determination outside of language, the most undetermined (١) unfolds in language as the most basic, abstract projection (1) for any kind of multitude, e.g., for a linear continuum.

However, since language is only a projection of a complementary notion, *1 is only real, but not reality*, i.e., it provides the foundation for multiple aspects of reality such as frequently encountered in consistent, abstract language:

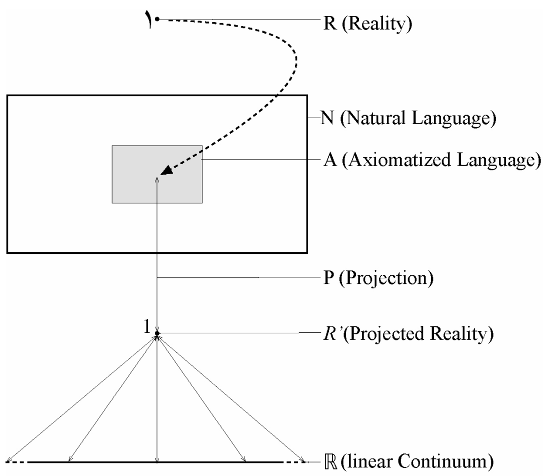

Figure 9 From a strict analytical perspective, it seems evident to first define the constituents of our abstract thought before extending them to multitudes, whether being points or numbers. On the other hand, we have to account for the process of language development which evidently works from naïve phenomenology to more and more differentiation only eventually leading to abstraction and unification.

If now the *axiom of reality* with its *total negation* or the equivalent *negation of totality* holds, it must not only provide our imagination with an intuitive idea about *what exists* and *how it exists* but it must also provide a prospective tool to deal with space as an *aspect of reality*.
It is in this sense, that our analytic tools, e.g., axiomatized language in association with natural meta-language, are only expected to be consistent whereas our hypothesis about what is *real* can at the most be subject to observables in order to be tried and tested (falsification).
Therefore, we can only presuppose *reality* according to our language capabilities while we associate the attribute "*real"* depending on our experience and its associated level of conformity, i.e., through empirical interaction in dedicated areas of application.
If however, a notion about *reality* would only be set equal to any kind of consensus with regard to which language to apply, e.g., *Euclidian Geometry* vs. *non-Euclidean Geometry* for our imagination about "space", or a consensus with respect to which conventions about physical units to apply, this would imply that *reality itself* would be subject to change concerning as well as both, *different languages* and a *consensus* about it whereas beforehand, the term "*reality*" is *per definitionem* independent of our subjective perception or imagination, i.e., "*reality*" is defined as being something *objective* vs. *subjective*, no matter what individually or collectively is considered to be *real*.
Two immediate physical aspects of this *reality* concern *quantum mechanical* phenomena. One is known as q*uantum entanglement* where *instant communication* between a *separated pair of photons* leads to the assumption of *entanglement*, also coined "*non-locality*" or "*non-spatial*" aspect of reality which requires a *deep revision* of our common intuition of *space topology* in a sense of *zero extension*. The other concerns an appropriate interpretation of a *relativistic quantum mechanical* expression known as "*Dirac equation"* $E = \pm\sqrt{p^2c^2 + m_0^2c^4}$ where what is called "matter" is unstable on principle with an implicit, infinitely-probable, total radiation-catastrophe (matter-antimatter annihilation) if considered as a single particle equation. Up-until today, only *asymmetric ad hoc hypothesis* in the *2nd quantization* of the *Dirac-field* with infinitely many additional particles assumed yield the self-evident conclusion that *the world exists*.

An ongoing effort will dedicate further publications to these physical aspects of reality as well as to set theoretical subjects.

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